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Scalar factors for non-canonical subgroup restriction of unitary group

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Abstract. A procedure has been outlined for obtaining the scalar factors (reduced Wigner coefficients) of the unitary group $U(nm) \supset U(n) \otimes U(m)$. This has been done at the permutation group level for $S_N \downarrow S_{N'} \otimes S_{N''}$ and the equality between the scalar factors of these two groups has been exploited. It has been shown that the scalar factors can be uniquely expressed in terms of the inner and outer product coupling coefficients of the permutation group. The ambiguity due to multiplicity is resolved at the level of the above coefficients.

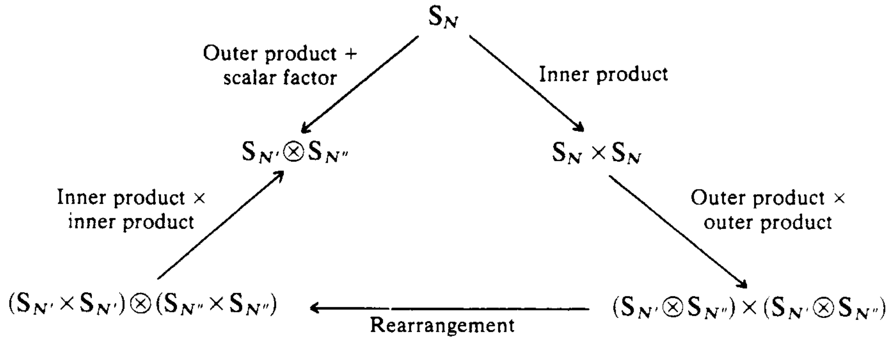
1. Introduction

The role of quark–quark interactions in providing a better understanding of the NN (De Rujula *et al* 1975, Neudatchin *et al* 1977, De Swart 1980) and NNN (Suzuki *et al* 1982) potentials is by now well established. The construction of six-quark and nine-quark wavefunctions in the colour–spin–isospin (CST) space has been carried out recently for specific representations of $U(12) \supset U(3) \otimes U(4)$ (Suzuki *et al* 1982, Obukhovsky *et al* 1982). These studies require, as a first step, the determination of the scalar factors (SF) for the restriction $U(nm) \supset U(n) \otimes U(m)$. For $SU(6) \supset SU(3) \otimes SU(2)$ the SF have been determined for the partition $q^N \to q^{N-1} \times q$ of an N-quark system (So and Strottman 1979, Strottman 1979). More general sets of SF for $q^N \to q^{N-3} \times q^3$ or, in general, $q^N \to q^{N'} \times q^{N''}$ have only been investigated to a limited extent (Suzuki *et al* 1982, Obukhovsky *et al* 1982, Chen 1981). There are two main reasons for the limited investigation of these SF in spite of the considerable importance attached to their determination. Firstly, there is a considerable multiplicity problem to be tackled arising from both the inner and outer product reductions of the representations of the group. The SF carry all these multiplicity labels and an unambiguous identification of their source is necessary before any computation. Secondly, the currently popular methods of determining the SF do not provide their direct workable definition. Thus in one of the approaches (Chen 1981) a set of cosets of the permutation group $S_{N'} \otimes S_{N''}$ in S_N has been diagonalised to obtain the SF. In another recent approach (Obukhovsky *et al* 1982), the SF for the CST group $SU(12) \supset SU(3) \otimes SU(2) \otimes SU(2)$ have been expressed in terms of the coefficients of fractional parentage (CFP) of SU(3) and SU(2) which are available. But this, in turn, is a specific case and the problem is liable to become complicated for a more general group such as $SU(8) \supset SU(4) \otimes SU(2)$ (Greenberg 1978). In spite of these limitations, however, one of the most interesting results of these investigations has been the realisation that the SF for the restriction $U(nm)\downarrow U(n)\otimes U(m)$ are identical to those for the restriction $S_N\downarrow S_{N'}\otimes S_{N''}$. This result follows from the duality between $U(n)$ and S_N investigated by Kramer and Seligman (1969a, b) and others. In the present note we will make use of this identity to work with the permutation group and obtain the necessary SF. We have been able to obtain an explicit usable expression for the SF occurring in the restriction $S_N\downarrow S_{N'}\otimes S_{N''}$ in terms of the Clebsch–Gordan coefficients (CGC) and the subduction coefficients (SC) of the permutation group. These coefficients are readily determinable using just the generators of S_N (Sarma 1981, Sahasrabudhe *et al* 1981) so that it is possible to determine the SF in a straightforward manner. In this process we have also been able to identify the multiplicity factor which leads to complications in SF determinations.

The main procedure is outlined in § 2 and a number of illustrative examples have been considered in § 3. A brief discussion of the method is presented in § 4.

2. Scalar factors for $S_N\downarrow S_{N'}\otimes S_{N''}$

Consider a system of N identical particles whose symmetry group is the permutation group S_N. We assume that the localisation of each particle requires two coordinate spaces (e.g. colour and spin–isospin) so that in terms of these the inner product symmetry group of the system is $S_N\times S_N$. We further partition the system into two subsystems such that $1, 2, \ldots, N'$ define the first subsystem and $N'+1, \ldots, N'+N''=N$ define the second. This implies that the basis spanning each S_N has to be adapted to the outer product subgroup $S_{N'}\otimes S_{N''}$. Under these circumstances the origin of the SF can be readily understood in terms of the diagram given below.



Following the steps outlined in this scheme, we now generate the SF for S_N under restriction to $S_{N'}\otimes S_{N''}$. Let $[\lambda]$, $[\mu]$, $[\nu]$ be three irreducible representations (irreps) of S_N and consider the product representation $[\mu]\times[\nu]$ of $S_N\times S_N$. The reduction of the product representation yields the Clebsch–Gordan series

$$[\mu]\times[\nu]=\sum_{\lambda} a^{\lambda}_{\mu\nu}[\lambda] \tag{1}$$

where $a^{\lambda}_{\mu\nu}$ is the multiplicity of occurrence of $[\lambda]$ in $[\mu]\times[\nu]$. If

$$\left|\begin{matrix}[\lambda]\tau_\lambda\\ i\end{matrix}\right\rangle,\ \left|\begin{matrix}[\mu]\\ j\end{matrix}\right\rangle,\ \left|\begin{matrix}[\nu]\\ k\end{matrix}\right\rangle$$

are the canonical Young orthogonal basis for S_N with τ_λ used to distinguish between multiply occurring $[\lambda]$, we have (Hamermesh 1964)

$$\left|\begin{matrix}[\lambda]\tau_\lambda\\ i\end{matrix}\right\rangle=\sum_j\sum_k C\begin{pmatrix}[\lambda]\tau_\lambda & [\mu] & [\nu]\\ i & j & k\end{pmatrix}\left|\begin{matrix}[\mu]\\ j\end{matrix}\right\rangle\left|\begin{matrix}[\nu]\\ k\end{matrix}\right\rangle \tag{2}$$

where the C-coefficients on the right are the standard CGC of S_N. Direct methods are now available for determining these coefficients (Schindler and Mirman 1977, Sahasrabudhe *et al* 1981). The next step in obtaining the SF is to consider the partitioning $N=N'+N''$. The basis states spanning the irreps $[\lambda']\otimes[\lambda'']$, $[\mu']\otimes[\mu'']$, $[\nu']\otimes[\nu'']$ of $S_{N'}\otimes S_{N''}$ can then be used to induce the irreps $[\lambda]$, $[\mu]$, $[\nu]$ respectively of S_N or *vice versa*. Before doing this let us first consider the CGC occurring in the reduction of $[\mu']\times[\nu']$ of $S_{N'}$ and $[\mu'']\times[\nu'']$ of $S_{N''}$:

$$\left|\begin{matrix}[\lambda']\tau_{\lambda'}\\ i'\end{matrix}\right\rangle=\sum_{j',k'} C\left(\begin{array}{c|cc}[\lambda']\tau_{\lambda'} & [\mu'] & [\nu']\\ i' & j' & k'\end{array}\right)\left|\begin{matrix}[\mu']\\ j'\end{matrix}\right\rangle\left|\begin{matrix}[\nu']\\ k'\end{matrix}\right\rangle, \tag{3}$$

$$\left|\begin{matrix}[\lambda'']\tau_{\lambda''}\\ i''\end{matrix}\right\rangle=\sum_{j'',k''} C\left(\begin{array}{c|cc}[\lambda'']\tau_{\lambda''} & [\mu''] & [\nu'']\\ i'' & j'' & k''\end{array}\right)\left|\begin{matrix}[\mu'']\\ j''\end{matrix}\right\rangle\left|\begin{matrix}[\nu'']\\ k''\end{matrix}\right\rangle, \tag{4}$$

where, as in (2), $\tau_{\lambda'}$ and $\tau_{\lambda''}$ are used to distinguish the multiply occurring $[\lambda']$, $[\lambda'']$ in $[\mu']\times[\nu']$ and $[\mu'']\times[\nu'']$ respectively. The outer product of the states defined by (3) and (4) then yields the basis states of $(S_{N'}\times S_{N'})\otimes(S_{N''}\times S_{N''})$ as

$$\left|\begin{matrix}[\lambda']\tau_{\lambda'}\\ i'\end{matrix}\right\rangle\otimes\left|\begin{matrix}[\lambda'']\tau_{\lambda''}\\ i''\end{matrix}\right\rangle=\sum_{\substack{j',k'\\ j'',k''}} C\left(\begin{array}{c|cc}[\lambda']\tau_{\lambda'} & [\mu'] & [\nu']\\ i' & j' & k'\end{array}\right)$$

$$\times C\left(\begin{array}{c|cc}[\lambda'']\tau_{\lambda''} & [\mu''] & [\nu'']\\ i'' & j'' & k''\end{array}\right)\left(\left|\begin{matrix}[\mu']\\ j'\end{matrix}\right\rangle\otimes\left|\begin{matrix}[\mu'']\\ j''\end{matrix}\right\rangle\right)\times\left(\left|\begin{matrix}[\nu']\\ k'\end{matrix}\right\rangle\otimes\left|\begin{matrix}[\nu'']\\ k''\end{matrix}\right\rangle\right). \tag{5}$$

We now restrict the outer products on the RHS of (5) to the product representation $[\mu]\times[\nu]$ of S_N so that (Sarma 1981, Kaplan 1975)

$$([\mu]\times[\nu])\downarrow\left(\left|\begin{matrix}[\lambda']\tau_{\lambda'}\\ i'\end{matrix}\right\rangle\otimes\left|\begin{matrix}[\lambda'']\tau_{\lambda''}\\ i''\end{matrix}\right\rangle\right)$$

$$=\sum_{\substack{j'j',j''k''\\ m,n}}\sum_{\tau_\mu,\tau_\nu} C\left(\begin{array}{c|cc}[\lambda']\tau_{\lambda'} & [\mu'] & [\nu']\\ i' & j' & k'\end{array}\right)C\left(\begin{array}{c|cc}[\lambda'']\tau_{\lambda''} & [\mu''] & [\nu'']\\ i'' & j'' & k''\end{array}\right)$$

$$\times S\left(\begin{array}{c|cc}[\mu]\theta_\mu & [\mu'] & [\mu'']\\ m & j' & j''\end{array}\right)S\left(\begin{array}{c|cc}[\nu]\theta_\nu & [\nu'] & [\nu'']\\ n & k' & k''\end{array}\right)\times\left|\begin{matrix}[\mu]\theta_\mu\\ m\end{matrix}\right\rangle\times\left|\begin{matrix}[\nu]\theta_\nu\\ n\end{matrix}\right\rangle \tag{6}$$

where θ_μ, θ_ν are indices used to distinguish the multiply occurring $[\mu]$, $[\nu]$ in $[\mu']\otimes[\mu'']$, $[\nu']\otimes[\nu'']$ respectively of $S_{N'}\otimes S_{N''}$ and the S-coefficients are the SC for the restriction $S_N\downarrow S_{N'}\otimes S_{N''}$.

Since the basis states on the RHS of (2) and (6) are the canonical ones spanning the irrep $[\mu]\times[\nu]$ of $S_N\times S_N$, we obtain the overlap between them as

$$\left\langle\begin{matrix}[\lambda]\tau_\lambda\\ i\end{matrix}\right|([\mu]\times[\nu])\downarrow\left(\left|\begin{matrix}[\lambda']\tau_{\lambda'}\\ i'\end{matrix}\right\rangle\otimes\left|\begin{matrix}[\lambda'']\tau_{\lambda''}\\ i''\end{matrix}\right\rangle\right)$$

$$=\sum_{jk,j'k',j''k''} C\left(\begin{array}{c|cc}[\lambda]\tau_\lambda & [\mu] & [\nu]\\ i & j & k\end{array}\right)C\left(\begin{array}{c|cc}[\lambda']\tau_{\lambda'} & [\mu'] & [\nu']\\ i' & j' & k'\end{array}\right)$$

$$\times C\left(\begin{array}{c|cc}[\lambda'']\tau_{\lambda''} & [\mu''] & [\nu'']\\ i'' & j'' & k''\end{array}\right) S\left(\begin{array}{c|cc}[\mu]\theta_\mu & [\mu'] & [\mu'']\\ j & j' & j''\end{array}\right)$$

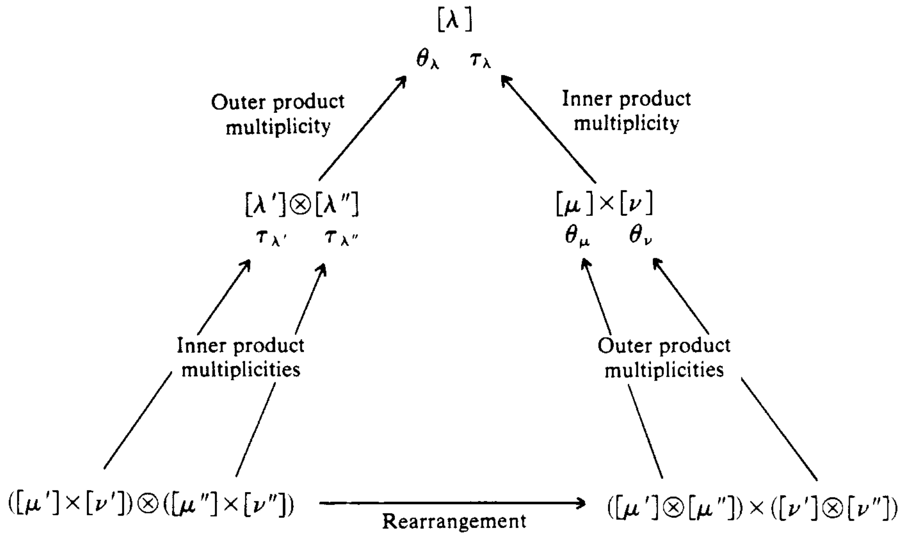
$$\times S\left(\begin{array}{c|cc}[\nu]\theta_\nu & [\nu'] & [\nu'']\\ k & k' & k''\end{array}\right). \tag{7}$$

In order to proceed to the final step of the scheme outlined in the beginning of this section, we need the definition of the SF through (Chen 1981)

$$\left|\begin{array}{c}[\lambda]\tau_\lambda\theta_\lambda\\ [\lambda']i'[\lambda'']i''\end{array}\right\rangle = \sum_{\substack{\mu',\mu''\\ \nu',\nu''}} \sum_{\substack{\theta_\mu,\theta_\nu\\ \tau_{\lambda'},\tau_{\lambda''}}} \left\{\begin{array}{c|cc}[\lambda]\tau_\lambda\theta_\lambda & [\mu]\theta_\mu & [\nu]\theta_\nu\\ [\lambda']\tau_{\lambda'}[\lambda'']\tau_{\lambda''} & [\mu'][\mu''] & [\nu'][\nu'']\end{array}\right\}$$

$$\times\left([\mu]\times[\nu]\downarrow\left|\begin{array}{c}[\lambda']\tau_{\lambda'}\\ i'\end{array}\right\rangle\left|\begin{array}{c}[\lambda'']\tau_{\lambda''}\\ i''\end{array}\right\rangle\right), \tag{8}$$

where the coefficients in braces on the right are the required scalar factors, and the various multiplicity labels are explained by the diagram below:



Using the definition given in (8) and the unitarity of SF (Chen 1981), we readily obtain the result

$$\sum_{\theta_\lambda}\left\{\begin{array}{cc|c}[\mu]\theta_\mu & [\nu]\theta_\nu & [\lambda]\tau_\lambda\theta_\lambda\\ [\mu'][\mu''] & [\nu'][\nu''] & [\lambda']\tau_{\lambda'}[\lambda'']\tau_{\lambda''}\end{array}\right\} S\left(\begin{array}{c|cc}[\lambda]\theta_\lambda & [\lambda'] & [\lambda'']\\ i & i' & i''\end{array}\right)$$

$$= \sum_{j,k,j',j'',k',k''\ P} C\left(\begin{array}{c|cc}[\lambda]\tau_\lambda & [\mu] & [\nu]\\ i & j & k\end{array}\right) C\left(\begin{array}{c|cc}[\lambda']\tau_{\lambda'} & [\mu'] & [\nu']\\ j' & k' & k''\end{array}\right)$$

$$\times C\left(\begin{array}{c|cc}[\lambda'']\tau_{\lambda''} & [\mu''] & [\nu'']\\ i'' & j'' & k''\end{array}\right) S\left(\begin{array}{c|cc}[\mu]\theta_\mu & [\mu'] & [\mu'']\\ j & j' & j''\end{array}\right) S\left(\begin{array}{c|cc}[\nu]\theta_\nu & [\nu'] & [\nu'']\\ k & k' & k''\end{array}\right). \tag{9}$$

From (9) we find that the index which can cause computational difficulties for SF is θ_λ which indicates the multiplicity of occurrence of $[\lambda]$ in $[\lambda']\otimes[\lambda'']$ in $S_N\downarrow S_{N'}\otimes S_{N''}$. If this occurrence is multiplicity free we find that the LHS of (9) reduces to a single term, and we obtain the result

$$\left\{\begin{matrix}[\mu]\theta_\mu & [\nu]\theta_\nu \\ [\mu'][\mu''] & [\nu'][\nu'']\end{matrix}\middle|\begin{matrix}[\lambda]\tau_\lambda \\ [\lambda']\tau_{\lambda'}[\lambda'']\tau_{\lambda''}\end{matrix}\right\} \equiv \frac{\text{RHS of 12}}{S\begin{pmatrix}[\lambda] & [\lambda'] & [\lambda''] \\ i & i' & i''\end{pmatrix}}. \tag{10}$$

The procedure for determining the SF is now evident if we start with definite irreps$[\mu]\times[\nu]$ of S_N and $[\lambda']\otimes[\lambda'']$ of $S_{N'}\otimes S_{N''}$. The inner and outer product series are then generated from these product representations. All $[\lambda]\subset([\mu]\times[\nu])\cap([\lambda']\otimes[\lambda''])$ are then selected. For the chosen pair $[\lambda']$, $[\lambda'']$ we then determine all possible $[\mu']\times[\nu']$, $[\mu'']\times[\nu'']$ of $S_{N'}$ and $S_{N''}$ respectively. From among these product representations only those $[\mu']$, $[\mu'']$ and $[\nu'][\nu'']$ are retained such that $[\mu']\otimes[\mu'']\supset[\mu]$ and $[\nu']\otimes[\nu'']\supset[\nu]$. The immediate check is that the number of irreps $[\lambda]$ (including multiplicity) should be equal to the number of product representations $([\mu']\otimes[\mu''])^{[\mu]}\times([\nu']\otimes[\nu''])^{[\nu]}$ since the SF matrix is unitary. For example, let $[\mu]=[2^3]$, $[\nu]=[2^2,1^2]$ be any two representations of S_6 and $[\lambda']=[2,1]$, $[\lambda'']=[2,1]$ be two representations of S_3. Then

$$[2^3]\times[2^2,1^2]=[2^2,1^2]+[3,2,1]+[3^2]+[4,1^2]+[5,1]$$

and

$$[2,1]\otimes[2,1]=[2^2,1^2]+[2^3]+[3,1^3]+2[3,2,1]+[3^2]+[4,1^2]+[4,2].$$

This leads to

$$\begin{aligned}[\lambda]&\supset([2^3]\times[2^2,1^2])\cap([2,1]\otimes[2,1])\\ &=\{[2^2,1^2],[3,2,1]_1,[3,2,1]_2,[3^2],[4,1^2]\}.\end{aligned}$$

The representations listed above thus label the rows of the SF matrix. Corresponding to these, the five column labels of the matrix follow on considering the outer product $[21]\otimes[21]$ of $S_3\otimes S_3$. Each of these $[21]$, in turn, can be expressed in terms of inner product irreps $[\mu']\times[\nu']$ of $S_3\times S_3$ as

$$[21]\supset\{[3]\times[21],[21]\times[3],[21]\times[21],[21]\times[1^3],[1^3]\times[21]\}.$$

A similar analysis for $[\lambda'']=[21]$ yields an identical set $[\mu'']\times[\nu'']$ of $S_3\times S_3$. Thus the product $[\lambda']\otimes[\lambda'']=[21]\otimes[21]$ can be expressed in terms of the set of irreps $([\mu']\times[\nu'])\otimes([\mu'']\times[\nu''])=([\mu']\otimes[\mu''])\times([\nu']\otimes[\nu''])$. From this set we select only those which induce $[\mu]=[2^3]$ from $[\mu']\otimes[\mu'']$ and $[\nu]=[2^21^2]$ from $[\nu']\otimes[\nu'']$. This yields the following product representations.

$$\begin{aligned}&([2^3]\times[2^2,1^2])\downarrow([2,1]\otimes[2,1])\\ &\quad=\{([2,1]\otimes[2,1])\times([2,1]\otimes[2,1]),([2,1]\otimes[2,1])\times([2,1]\otimes[1^3]),\\ &\qquad([2,1]\otimes[2,1])\times([1^2]\otimes[2,1]),([2,1]\otimes[2,1])\times([1^3]\otimes[1^3]),\\ &\qquad([1^3]\otimes[1^3])\times([2,1]\otimes[2,1])\}.\end{aligned}$$

These five product representations thus label the columns of the SF matrix. Having determined the possible $[\lambda]$ in this manner, we first choose a $|{}^{\lambda}_{i}\rangle$ which has a subtableau structure corresponding to a given $|{}^{\lambda'}_{i'}\rangle$ over the first N' particles. We then determine the inner product (CG) coefficients for all possible $[\mu']\times[\nu']$ and $[\mu'']\times[\nu'']$ leading respectively to $|{}^{\lambda'}_{i'}\rangle$, $|{}^{\lambda''}_{i''}\rangle$ with the $[\mu']$, $[\nu']$, $[\mu'']$, $[\nu'']$ chosen such that they lead to fixed $[\mu]$, $[\nu]$ of S_N. The SC for $[\mu]\downarrow[\mu']\otimes[\mu'']$ and $[\nu]\downarrow[\nu']\otimes[\nu'']$ are also determined.

Using the product state as obtained in (6), the overlap with the fixed $|{}^{\lambda}_{i}\rangle$ is determined, leading to the result as on the RHS of (7); we then determine the SC for $[\lambda]\downarrow[\lambda']\otimes[\lambda'']$ and determine the overlap with $|{}^{\lambda}_{i}\rangle$ as on the RHS (9). Depending on whether this subduction is multiplicity dependent $(([\lambda,]\downarrow[\lambda']\otimes[\lambda''])>1)$ or multiplicity free, (9) or (10) is used to determine the SF. In § 3 we illustrate the working of this scheme using a number of examples.

3. Illustrative examples

We first generate the SF matrix for a nine-quark system studied by Suzuki *et al* (1982) for the group–subgroup chain $SU(12)\supset SU(3)\otimes SU(4)$ in the CST space for the partition $9=6+3$. The corresponding permutation group chain we need to consider is $S_9\supset S_6\otimes S_3$. One of the tables (table 5, Suzuki *et al* 1982) deals with the representations $[\mu]$, $[\nu]$, $[\lambda']$, $[\lambda'']$ corresponding to $[3^3]$, $[7,1^2]$, $[2^2,1^2]$, $[1^3]$ respectively. Using the analysis of § 2 we find that only the representations $[\lambda]=[3,2^3]$, $[3^2][1^3]$, $[3^2,2,1]$ of S_9 need to be considered. These representations are multiplicity free in $[2^2,1^2]\otimes[1^3]$ of $S_6\otimes S_3$ so that (10) may be used. The possible product representations $([\mu']\otimes[\mu''])^{[\mu]}([\nu']\otimes[\nu''])^{[\nu]}$ of $(S_6\otimes S_3)\times(S_6\otimes S_3)\subset(S_9\times S_9)$ are $([2^3]\otimes[1^3])^{[3^3]}\times([5,1]\otimes[3])^{[7,1^2]}$, $([2^3]\otimes[1^3])^{[3^3]}\times([4,1^2]\otimes[3])^{[7,1^2]}$ and $([321]\otimes[2,1])^{[3^3]}\times([5,1]\otimes[2,1])^{[7,1^2]}$. As an illustration we generate below the row of the SF matrix corresponding to $([2^3]\otimes[1^3])^{[3^3]}\times([5,1]\otimes[3])^{[7,1^2]}$. For $[3,2^3]\downarrow[2^2,1^2]\otimes[1^3]$, the reference basis states chosen are

$$\left|{\lambda \atop i}\right\rangle=\left|{[3,2^3] \atop (123412341)}\right\rangle,\qquad \left|{\lambda' \atop i'}\right\rangle=\left|{[2^2,1^2] \atop (123412)}\right\rangle,\qquad \left|{\lambda'' \atop i''}\right\rangle=\left|{[1^3] \atop (123)}\right\rangle,$$

so that the SC is non-zero. In the above the lower entries are lattice permutation symbols defined in terms of standard Young tableaux as

$$(123412341)=\begin{matrix}159\\26\\37\\48\end{matrix}$$

etc. Since the lattice permutation symbol also defines the Young diagram corresponding to the given representation, we will henceforth avoid mentioning the representation explicitly in the basis kets. The CGC for $[3,2^3]\subset[3^3]\times[7,1^2]$ can be readily determined (Sahasrabudhe *et al* 1981, Schindler and Mirman (1977) so that

$$\begin{aligned}|(123412341)\rangle=(5/12\sqrt{21})\{&\tfrac{24}{5}|([1123]23123)\rangle|([1211]11131)\rangle\\&-\tfrac{8}{5}\sqrt{3}|([1123]21323)\rangle|([1211]11131)\rangle+\tfrac{4}{5}\sqrt{6}|([1123]23123)\rangle\\&\times|(1211]11131)\rangle-\tfrac{21}{5}|([1123]12233)\rangle|([1211]11311)\rangle\\&+\tfrac{7}{5}\sqrt{3}|([423]21233)\rangle|([1211]11311)\rangle\\&+3\sqrt{7/5}|([1123]12233)\rangle|([1211]13111)\rangle\\&+\tfrac{7}{5}\sqrt{21/5}|([1123]12233)\rangle|([1211]11311)\rangle\\&-\tfrac{12}{5}\sqrt{14/5}|([1123]21233)\rangle|([1211]31111)\rangle\end{aligned}$$

$$-\sqrt{3}|([1123]12323)\rangle|([1211]11311)\rangle$$
$$+|([1123]21323)\rangle|([1211]11311)\rangle$$
$$+\tfrac{8}{5}\sqrt{2}|([1123]23123)\rangle|([1211]11311)\rangle$$
$$-\tfrac{4}{5}\sqrt{14/5}|([1123]23123)\rangle|([1211]13111)\rangle$$
$$-\tfrac{4}{5}\sqrt{21/5}|([1123]23123)\rangle|([1211]31111)\rangle$$
$$-\sqrt{21/5}|([1123]12323)\rangle|([1211]13111)\rangle$$
$$-\tfrac{7}{5}\sqrt{7/5}|([1123]21323)\rangle|([1211]13111)\rangle$$
$$-\tfrac{4}{5}\sqrt{42/5}|([1123]21323)\rangle|([1211]31111)\rangle\} \tag{11}$$

where for compactness we have introduced the square brackets in each lattice permutation symbol to define products such as

$$([1123]12233)\rangle|([1211]13111)\rangle=(1/\sqrt{3})|(112312233)\rangle|(121113111)\rangle$$
$$-|(121312233)\rangle|(112113111)\rangle+|(123112233)\rangle|(111213111)\rangle. \tag{12}$$

Similarly,

$$|(123412)\rangle=|([1223]23)\rangle|([1211]11)\rangle \tag{13}$$

and

$$|(123)\rangle=|(123)\rangle|(111)\rangle. \tag{14}$$

Using the procedures for outer product reduction (Kaplan 1975, Sarma 1981) and the results of (13), (14), we obtain the result

$$(|(123412)\rangle\otimes|(123)\rangle)^{[3^3]\times[71^2]}$$
$$=([3^3]\downarrow|([1123]23)\rangle\otimes|(123)\rangle)\times([71^2]\downarrow([1211]11)\rangle\otimes|(111)\rangle)$$
$$=(1/2\sqrt{7})|([1123]23123)\rangle\times\{\sqrt{7}|([1211]11113)\rangle$$
$$+3|([1211]11131)\rangle+2\sqrt{3}|([1211]11311)\rangle\}. \tag{15}$$

The overlap of (11) and (15) yields the RHS of (9) as

$$\langle(123412341)|(|123412)\rangle\otimes|(123)\rangle)^{[3^3]\times[7,1^2]}\rangle=1/3\sqrt{2}. \tag{16}$$

On the other hand,

$$(|(123412)\rangle\otimes|(123)\rangle)^{[3,2^3]}=(1/3\sqrt{2})\{\sqrt{3}|(123412341)\rangle$$
$$-\sqrt{5}|(1234123414)\rangle+\sqrt{10}|(123412134)\rangle\}$$

so that the required SC is

$$S\begin{pmatrix}[32^3] & [2^21^2] & [1^3]\\ (123412341) & (123412) & (123)\end{pmatrix}=\frac{1}{\sqrt{6}}. \tag{17}$$

Thus the multiplicity free SF resulting from (10) is

$$\left\{\begin{matrix}[3,2^3]\\ [2^2,1^2][1^3]\end{matrix}\middle|\begin{matrix}[3^3] & [7,1^2]\\ [2^3][1^3] & [5,1][3]\end{matrix}\right\}=\frac{\sqrt{6}}{3\sqrt{2}}=\frac{1}{\sqrt{3}}. \tag{18}$$

For the representation $[3^2, 1^3]$ of S_9 a similar procedure can be readily carried out.

The CGC for this representation can be obtained as before in terms of the basis spanning $[3^3]\times[7,1^2]$. The product states spanning $[3^3]\times[7,1^2]$ induced from $([2^3]\otimes[1^3])^{[3^3]}\times([5,1]\otimes[3])^{[7,1^2]}$ are the same as on the RHS of (15). The overlap factor can thus be determined. The required SC can be determined as outlined earlier so that

$$\left\{\begin{matrix}[3^2,1^3] \\ [2^2,1^2][1^3]\end{matrix}\middle|\begin{matrix}[3^3] & [7,1^2] \\ [2^3][1^3] & [5,1][3]\end{matrix}\right\} = \frac{(\text{overlap factor})}{S\begin{pmatrix}[3^2,1^3] \\ (123412512)\end{pmatrix.\left|\begin{matrix}[2^2,1^2] & [1^3] \\ (123412) & (123)\end{matrix}\right)}$$

$$= (3/\sqrt{2})\times(2/3\sqrt{3}) = \sqrt{2/3}. \tag{19}$$

Similar analysis for the representation $[3^2,2,1]$ yields zero SF so that we obtain the first row of the SF matrix as

$$\left.([2^3]\otimes[1^3])^{[3^3]}\times([5,1]\times[3])^{[7,1^2]}\right| \frac{[3,2^3]}{1/\sqrt{3}}\ \frac{[3^2,1^3]}{\sqrt{2}/\sqrt{3}}\ \frac{[3^2,2,1]}{0}.$$

We find that the result agrees both in sign and magnitude with the reduced Wigner coefficients for $SU(12)\supset SU(3)\otimes SU(4)$ in the partition $q^9\to q^6\times q^3$ obtained by Suzuki *et al* (1982).

As a final example we have generated the complete five-dimensional SF matrix for the representations $[\mu]\equiv[2^3]$, $[\nu]\equiv[2^2,1^2]$ of S_6 induced from $[\lambda']\equiv[2,1]$ and $[\lambda'']\equiv[2,1]$ of $S_3\otimes S_3$. The columns of the SF matrix have been labelled by the product representations

$$([2,1]\otimes[2,1])^{[2^3]}\times([2,1]\otimes[2,1])^{[2^2,1^2]}([2,1]\otimes[2,1])^{[2^3]}([2,1]\otimes[1^3])^{[2^2,1^2]},$$

$$([2,1]\otimes[2,1])^{[2^3]}\times([1^3]\otimes[2,1])^{[2^2,1^2]},([2,1]\otimes[2,1])^{[2^3]}$$

$$\times([1^3]\otimes[1^3])^{[2^2,1^2]},\ ([1^3]\otimes[1^3])^{[2^3]}([2,1]\otimes[2,1])^{[2^2,1^2]}$$

and the rows by $[2^2,1^2]$, $[3^2]$, $[4,1^2]$, $[3,2,1]_1$, $[3,2,1]_2$. We find that the first three rows are multiplicity free and can be readily determined. The necessary CGC for each $[\lambda]$ of S_6 have been listed in table 1. The induced product representations are summarised in table 2. Using these, the necessary overlap factor can be obtained. In addition we require also the SC for $[\lambda]\downarrow[2,1]\otimes[2,1]$. These have been presented in table 3. Using the results of these tables and (10) we can easily determine the first three rows of the SF matrix listed in table 4. The only case of multiplicity is the representation $[3,2,1]$ of S_6 which occurs twice in $[2,1]\otimes[2,1]$. In this case the overlap factors are still as determined from the appropriate rows of tables 1 and 2 but the LHS of (9) is now a sum over two terms. Noting that this occurrence is due to the outer product multiplicity, we can shift the ambiguity in determining the SF to the SC. Thus, by choosing a set of SC which have non-zero values for only one state $\left|{[3,2,1]1 \atop i}\right\rangle$ we can again use (10) and determine the SF as for the multiplicity free case. The SF for $[3,2,1]2$ can then be readily determined by using row orthogonality at the SF matrix and normalisàtion of the basis. The results obtained are presented in the last two rows of table 4. If the multiplicity happens to be >2, one of the rows of the SF matrix for this $[\lambda]$ can be determined as in the present case. For the other rows, the row and column orthogonalities of the unitary SF matrix can be used to determine the required quantities.

Table 1. Inner product expansion of basis states $|{}^{\lambda}_{i}\rangle$ of S_6 in terms of product states spanning $[\mu]\times[\nu]$ expressed in terms of Lattice permutation symbols.

$$\begin{aligned}
|(121311)\rangle = {} & |(112233)\rangle\times\{-(1/2\sqrt{3})|(121342)\rangle \\
& -(1/2\sqrt{6})|(121324)\rangle+(1/\sqrt{6})|(123142)\rangle\}+|(112323)\rangle \\
& \times\{(1/2\sqrt{6})|(121234)\rangle-\tfrac{1}{3}|(123124)\rangle+(1/3\sqrt{2})|(123142)\rangle\} \\
& +|(121233)\rangle\times\{-(1/2\sqrt{6})|(112324)\rangle-(1/2\sqrt{3})|(112342)\rangle\} \\
& +(1/2\sqrt{6})|(121323)\rangle\times|(112234)\rangle \\
& +|(123123)\rangle\times\{-(1/2\sqrt{3})|(112234)\rangle+\tfrac{1}{3}|(112324)\rangle \\
& -(1/3\sqrt{2})|(112342)\rangle\}
\end{aligned}$$

$$\begin{aligned}
|(121122)\rangle = {} & |(112233)\rangle\times\{(\sqrt{5}/6\sqrt{6})|(121324)\rangle+(5/6\sqrt{3})|(121342)\rangle \\
& +(\sqrt{5}/6\sqrt{3})|(123124)\rangle+(\sqrt{5}/3\sqrt{6})|(123142)\rangle\} \\
& +|(121233)\rangle\{(\sqrt{5}/6\sqrt{6})|(112324)\rangle+(\sqrt{5}/6\sqrt{3})|(112342)\rangle\} \\
& +|(112323)\rangle\times\{(\sqrt{5}/6\sqrt{6})|(121234)\rangle-(\sqrt{10}/9)|(121324)\rangle \\
& +(\sqrt{5}/9)|(121342)\rangle+(\sqrt{5}/9)|(123124)\rangle-(\sqrt{5}/9\sqrt{2})|(123142)\rangle \\
& -(2/3\sqrt{6})|(123412)\rangle\}+|(121323)\rangle\times\{-(\sqrt{10}/9)|(112324)\rangle \\
& +(\sqrt{5}/6\sqrt{6})|(112234)\rangle+(\sqrt{5}/9)|(112342)\rangle\}+|(123123)\rangle \\
& \times\{(\sqrt{5}/6\sqrt{3})|(112234)\rangle+(\sqrt{5}/9)|(112324)\rangle-(\sqrt{5}/9\sqrt{2})|(112342)\rangle\}
\end{aligned}$$

$$\begin{aligned}
|(121342)\rangle = {} & |(112323)\rangle\times\{-(2/3\sqrt{10})|(112234)\rangle-(2/\sqrt{15})|(123124)\rangle\} \\
& +|(112233)\rangle\times\{(2/3\sqrt{10})|(121324)\rangle-(2/3\sqrt{5})|(123124)\rangle\} \\
& -(2/3\sqrt{10})|(121323)\rangle\times|(112234)\rangle \\
& +|(123123)\rangle\times\{(2/3\sqrt{5})|(112234)\rangle+(2/\sqrt{15})|(112324)\rangle\} \\
& +(2/3\sqrt{10})|(121233)\rangle+|(112324)\rangle
\end{aligned}$$

$$\begin{aligned}
|(121321)\rangle = {} & |(112233)\times\{-(1/\sqrt{10})|(121324)\rangle+(1/\sqrt{5})|(123124)\rangle \\
& -(1/4\sqrt{5})|(121342)\rangle+(1/2\sqrt{10})|123142)\rangle\}+|(121233)\rangle \\
& \times\{-(1/\sqrt{10})|(112324)\rangle-(1/4\sqrt{5})|(112342)\rangle\}+|(112323)\rangle \\
& \times\{-(1/\sqrt{10})|(121234)\rangle-(\sqrt{3}/2\sqrt{10})|(123142)\rangle\}-(1/\sqrt{10})(121323)\rangle \\
& \times(112234)\rangle+|(123123)\rangle\times\{(1/\sqrt{5})|(112234)\rangle+(\sqrt{3}/2\sqrt{10})|(112342)\rangle\}
\end{aligned}$$

4. Discussion

The procedure outlined in § 2 is computationally feasible. There are basically three stages in implementing it. Firstly, we require the CGC for one state $|{}^{[\lambda]}_{i}\rangle$ of S_N in terms of the basis spanning $[\mu]\times[\nu]$. The procedure for determining these coefficients is now readily avaliable (Schindler and Mirman 1977, Sahasrabudhe *et al* 1981). The second stage of the procedure involves the determination of the SC leading to $|{}^{[\lambda]}_{i}\rangle$ from states spanning $[\lambda']\otimes[\lambda'']$ and similar ones for $[\mu]$ and $[\nu]$ induced from $[\mu']\otimes[\mu'']$ and $[\nu']\otimes[\nu'']$ respectively. The procedures for determining these coefficients are relatively simple (Kaplan 1975, Sarma 1981). The final stage involves use of (9) or (10). For the multiplicity free case this is relatively simple. For two-fold multiplicity, the SC can be chosen suitably so that the multiplicity free approach can be used. For a three or higher-fold multiplicity we can use the fact that the SF matrix is unitary to generate the necessary quantities.

The procedure is also interesting in that it clearly brings out the origin of multiplicity factors which can cause computational difficulties. This is an advantage since, as illustrated in § 3, we can shift the ambiguity to the SC from the SF and simplify the calculations to some extent.

Table 2. Basis states of $[2^3]\times[2^2,1^2]$ of S_6 induced from $([\mu']\otimes[\mu''])\times([\nu']\otimes[\nu''])$ of $(S_3\otimes S_3)\times(S_3\otimes S_3)$.

$$|([2,1]\otimes[2,1])^{[2^3]}\times([1,1]\otimes[2,1])^{[2^2,1^2]}\rangle\rangle$$
$$=|(112233)\rangle\rangle\times\{\tfrac{1}{4}|(121234)\rangle\rangle+(1/4\sqrt{3})|(121324)\rangle\rangle-(1/\sqrt{6})|(121342)\rangle\rangle\}$$
$$+|(112323)\rangle\rangle\times\{(1/4\sqrt{3})|(121234)\rangle\rangle+\tfrac{5}{12}|(121324)\rangle\rangle+(1/3\sqrt{2})|(121342)\rangle\rangle\}$$
$$+|(121233)\rangle\rangle\times\{\tfrac{1}{4}|(112234)\rangle\rangle+(1/4\sqrt{3})|(112324)-(1/\sqrt{6})|(112342)\rangle\rangle\}$$
$$+(121323)\rangle\rangle\times\{(1/4\sqrt{3})|(112234)\rangle\rangle+\tfrac{5}{12}|(112324)\rangle\rangle+(1/3\sqrt{2})|(112342)\rangle\rangle\}$$
$$|([2,1]\otimes[2,1])^{[2^3]}\times([1^3]\otimes[2,1])^{[2^2,1^2]}\rangle$$
$$=|(112233)\rangle\rangle\times\{-(1/\sqrt{6})|(123124)\rangle\rangle-(1/4\sqrt{3})|(123142)\rangle\rangle+(\sqrt{5}/4)(123412)\rangle\rangle\}$$
$$+|(112323)\rangle\rangle\times\{(1/3\sqrt{2})|(123124)\rangle\rangle+\tfrac{7}{12}|(123142)\rangle\rangle+(\sqrt{5}/4\sqrt{3})|(123412)\rangle\rangle\}$$
$$|([2,1]\otimes[2,1])^{[2^3]}\times([2,1]\otimes[1^3])^{[2^2,1^2]}\rangle$$
$$=|(112233)\rangle\rangle\times\{\tfrac{1}{2}|(121234)\rangle\rangle-(1/2\sqrt{3})|(121324)\rangle\rangle+(1/2\sqrt{6})|(121342)\rangle\rangle\}$$
$$+|(112323)\rangle\rangle\times\{-(1/2\sqrt{3})|(121234)\rangle\rangle+\tfrac{1}{6}|(121324)\rangle\rangle-(1/6\sqrt{2})|(121342)\rangle\rangle$$
$$+|(121233)\rangle\rangle\times\{\tfrac{1}{2}|(112234)\rangle\rangle-(1/2\sqrt{3})|(112324)\rangle\rangle+(1/2\sqrt{6})|(112342)\rangle\rangle\}$$
$$+|(121323)\rangle\rangle\times\{-(1/2\sqrt{3})|(112234)\rangle\rangle+\tfrac{1}{6}|(112324)\rangle\rangle-(1/6\sqrt{2})|(112342)\rangle\rangle\}$$
$$|([2,1]\otimes[2,1])^{[2^3]}\times([1^3]\otimes[1^3])^{[2^2,1^2]}\rangle\rangle$$
$$=|(112233)\rangle\rangle\times\{-(\sqrt{5}/2\sqrt{3})|(123124)+(\sqrt{5}/2\sqrt{6})|(123142)\rangle\rangle-(1/2\sqrt{2})|(123412)\rangle\rangle\}$$
$$+|(112323)\rangle\rangle\times\{(\sqrt{5}/6)|(123124)\rangle\rangle-(\sqrt{5}/6\sqrt{2})|(123142)\rangle\rangle+(1/2\sqrt{6})|(123412)\rangle\rangle\}$$
$$|([1^3]\times[1^3])^{[2^3]}\times([2,1]\times[2,1])^{[2^2,1^2]}\rangle\rangle$$
$$=|(123123)\rangle\rangle\times\{-(1/2\sqrt{3})|(112234)\rangle\rangle+\tfrac{1}{6}|(112324)\rangle\rangle+(2\sqrt{2}/3)|(112342)\rangle\rangle\}$$

Table 3. Subduction coefficient occurring in the restriction $[\lambda]\downarrow[2,1]\times[2,1]$ of $S_6\downarrow S_3\otimes S_3$ where $[\lambda]=[4,1^2]$, $[3^2]$, $[2^2,1^2]$, $[3,2,1]1$, $[3,2,1]2$.

$[\lambda]$		$\|(112)\rangle\rangle\otimes\|(112)\rangle\rangle$	$\|(112)\rangle\rangle\otimes\|(121)\rangle\rangle$
$[4,1^2]$	$\|(112113)\rangle\rangle$	$-2/\sqrt{5}$	0
	$\|(112131)\rangle\rangle$	$\sqrt{3}/2\sqrt{10}$	$-\sqrt{5}/2\sqrt{2}$
	$\|(112311)\rangle\rangle$	$-1/2\sqrt{2}$	$-\sqrt{3}/2\sqrt{2}$
$[3^2]$	$\|(112122)\rangle\rangle$	$-1/2$	$-\sqrt{3}/2$
	$\|(112212)\rangle\rangle$	$-\sqrt{3}/2$	$1/2$
$[2^2,1^2]$	$\|(112234)\rangle\rangle$	$1/2$	$-1/2\sqrt{3}$
	$\|(112324)\rangle\rangle$	$\sqrt{3}/2$	$1/6$
	$\|(112342)\rangle\rangle$	0	$2\sqrt{2}/3$
$[3,2,1]1$	$\|(112123)\rangle\rangle$	$-1/\sqrt{6}$	$1/2\sqrt{2}$
	$\|(112132)\rangle\rangle$	$1/2\sqrt{2}$	0
	$\|(112213)\rangle\rangle$	$-1/\sqrt{2}$	$1/2\sqrt{3}$
	$\|(112231)\rangle\rangle$	0	$-\sqrt{5}/2\sqrt{2}$
	$\|(112312)\rangle\rangle$	$\sqrt{5}/2\sqrt{6}$	0
	$\|(112321)\rangle\rangle$	0	$\sqrt{5}/2\sqrt{6}$
$[3,2,1]2$	$\|(112123)\rangle\rangle$	$-\sqrt{5}/4\sqrt{6}$	$-\sqrt{5}/2\sqrt{2}$
	$\|(112132)\rangle\rangle$	$-\sqrt{5}/4\sqrt{2}$	$\sqrt{15}/4\sqrt{2}$
	$\|(112213)\rangle\rangle$	$-\sqrt{5}/4\sqrt{2}$	$\sqrt{5}/4\sqrt{6}$
	$\|(112231)\rangle\rangle$	$3/4\sqrt{6}$	$-1/4\sqrt{2}$
	$\|(112312)\rangle\rangle$	$-5/4\sqrt{6}$	$3/4\sqrt{2}$
	$\|(112321)\rangle\rangle$	$3/4\sqrt{2}$	$-1/4\sqrt{6}$

Table 4. Isoscalar factors for $[2^3]\times[2^21^2]$ of S_6 under the restriction $[\lambda]\downarrow[\lambda_1]\otimes[\lambda_2]=[21]\otimes[21]$ of $S_3\otimes S_3$.

$(\mu_1\mu_2)(\nu_1\nu_2)\rightarrow$ $[\lambda]$	[21][21] [21][21]	[21][21] $[1^3]$[21]	[21][21] [21]$[1^3]$	[21][21] $[1^3][1^3]$	$[1^3][1^3]$ [21][21]
$[41^2]$	$2/3$	$-1/3$	$-1/3$	$-\sqrt{10}/6$	$-\sqrt{2}/6$
$[3^2]$	$-\sqrt{10}/6$	$-\sqrt{10}/6$	$-\sqrt{10}/6$	$1/6$	$-\sqrt{5}/6$
$[2^21^2]$	$\sqrt{10}/10$	$-\sqrt{10}/10$	$-\sqrt{10}/10$	$1/2$	$3\sqrt{20}/20$
$[321]_1$	$-\sqrt{15}/45$	$-8\sqrt{15}/45$	$7\sqrt{15}/45$	$-\sqrt{6}/9$	$2\sqrt{30}/45$
$[321]_2$	$-2\sqrt{3}/9$	$\sqrt{3}/9$	$-2\sqrt{3}/9$	$-2\sqrt{30}/9$	$2\sqrt{6}/9$

In spite of the large number of examples considered in § 3, the procedure has inherent limitations which need to be brought out. The major one is that extensive tables of SF cannot be prepared since the procedure has not been programmed. As already mentioned, a number of stages are involved and each stage could require quite complicated logic. At the same time, any specific set of SF required could be readily generated without going to a computer. It is this limitation which shows up as a disadvantage when compared with other methods such as those of Suzuki *et al* (1982) or Obukhovsky *et al* (1982). In spite of some of the obvious advantages of these methods over ours, it should be emphasised that they are tied up with either the knowledge of the Wigner and Racah coefficients of SU(3) or the CFP. Thus a direct extension to general $SU(nm)\supset SU(n)\otimes SU(m)$ does not appear feasible, unlike in the present procedure which is completely independent of the orders n, m.

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